

## Poisson Distributions (As a limiting case of Binomial Distribution)

- Poisson distribution was derived in 1837 by a French mathematician Simeon D. Poisson.
- Poisson distribution may be obtained as a limiting case of Binomial probability distributions under the following conditions:
  - (i)  $n$ , the number of trials is indefinitely large i.e  $n \rightarrow \infty$
  - (ii)  $p$ , the constant probability of success for each trial is definitely small i.e  $p \rightarrow 0$ .
  - (iii)  $np = m$  is finite.

Thus Poisson distribution

$$P(r) = P(X=r) = \frac{e^{-m} \cdot m^r}{r!}, r=0, 1, 2, 3, \dots$$

where,  $X$  is the number of successes (occurrences of the event),  $m = np$  and

$$e = 2.71828 \quad [\text{Natural logarithms}]$$

$$r! = r(r-1)(r-2) \cdots \times 3 \times 2 \times 1$$

Importance of Poisson Distribution :

- It can be used to explain the behaviour of the discrete random variables.
- Also useful in the field of Queuing Theory (waiting time problems), Insurance, Physics, Biology, Business, Economics and Industry.

→ Most of the Temporal Distributions (dealing with events which are supposed to occur in equal intervals of time) and spatial distributions (dealing with events which are supposed to occur in intervals of equal length along a straight line) follow the Poisson probability law.

$r$	$P(r)$	$rP(r)$	$r^2 P(r)$
0	$e^{-m}$	0	0
1	$me^{-m}$	$1 \cdot me^{-m}$	$me^{-m}$
2	$\frac{m^2 e^{-m}}{2!}$	$\frac{2 \cdot m^2 e^{-m}}{2!}$	$\frac{2^2 m^2 e^{-m}}{2!}$
3	$\frac{m^3 e^{-m}}{3!}$	$\frac{3 \cdot m^3 e^{-m}}{3!}$	$\frac{3^2 m^3 e^{-m}}{3!}$

$m \rightarrow$  parameter of the poisson distribution.

$$\text{Mean} = \sum_{r=0}^{\infty} r P(r)$$

$$= me^{-m} + 2 \frac{m^2 e^{-m}}{2!} + 3 \cdot \frac{m^3 e^{-m}}{3!} + \dots$$

$$= me^{-m} \left[ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right]$$

$$= me^{-m} \times e^m$$

$$= me^{-m+m}$$

$$= me^0 \quad [e^0 = 1]$$

$$= m$$

$$\text{Variance} = \sum r^2 P(r) - [\sum r P(r)]^2$$

$$= \sum r^2 P(r) - (\text{mean})^2$$

$$= \sum r^2 P(r) - (m)^2$$

$$\begin{aligned}
 E(r^2) P(r) &= m e^{-m} + 2^2 \cdot \frac{m^2 e^{-m}}{2!} + 3^2 \frac{m^3 e^{-m}}{3!} + \dots \\
 &= m e^{-m} \left[ 1 + 2m + \frac{3}{2!} m^2 + \frac{4}{3!} m^3 + \dots \right] \\
 &= m e^{-m} \left[ \left\{ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right\} + \left\{ m + \frac{2m^2}{2!} + \frac{3m^3}{3!} + \dots \right\} \right] \\
 &= m e^{-m} \left[ \left\{ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right\} + m \left\{ 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \right\} \right] \\
 &= m e^{-m} [e^m + m e^m] \\
 &= m e^{-m} \cdot e^m + e^m (1+m) \\
 &= m(1+m) e^0 \\
 &= m(1+m)
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{variance} &= m(1+m) - m^2 \\
 &= m + m^2 - m^2 \\
 &= m.
 \end{aligned}$$

Hence for the poison distribution  $m = \text{mean} = \text{variance}$

Reference : Fundamentals of Statistics by S.C. Gupta.